# LATTICE QCD and the CKM MATRIX

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|---|---|---|---|
| Mon | WHAT | is lattice gauge theory ? | ASK |
| Tues | WHY | should you believe it ? | ASK |
| Wed | HOW | are calculations done ? | MDP |
| Thurs | WHEN | will uncertainties be sub % ? | PBM |

A timely week to give these lectures:

| | | |
|---|---|---|
| 1987 | ACPMAPS | started |
| 1991 | " | upgraded 600 x 40 MHz |
| May 15, 2002 | " | decommissioned |

# WHAT is QCD?

A non-abelian gauge theory $G = SU(3)$

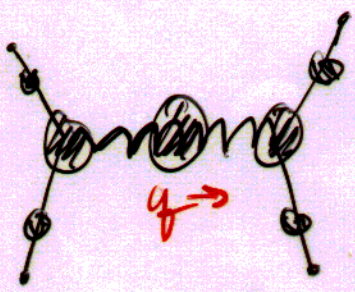
$$\mathcal{L} = \frac{1}{4g_0^2} \sum_{a=1}^{8} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{f=1}^{n_f} \bar{q}_f (\not{D} + m_f) q_f$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$$ gluons

$q, \bar{q}$ quarks anti-quarks

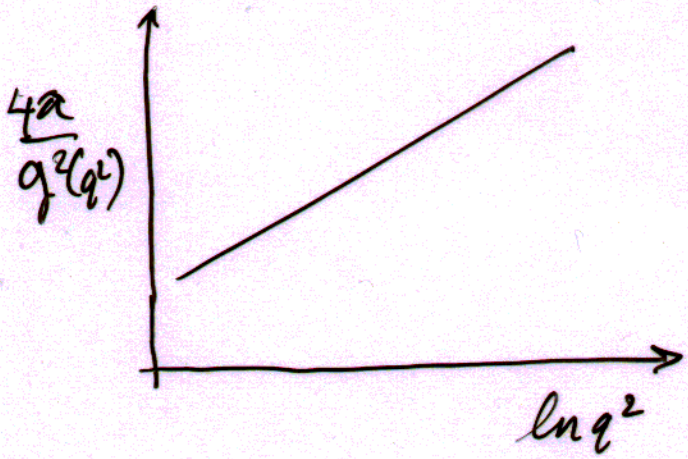
$8 = 3^2 - 1$

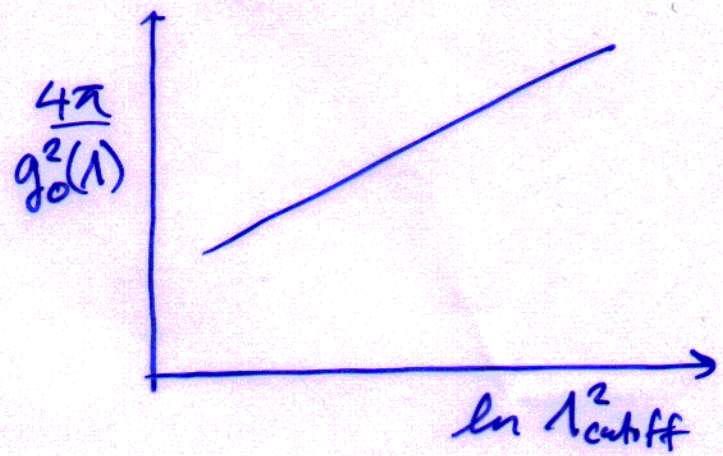
A signature feature of QCD is asymptotic freedom:




$$\sim \frac{g_0^2(\Lambda)}{4\pi} + \beta_0 \frac{g_0^4(\Lambda)}{(16\pi^2)} \ln \frac{q^2}{\Lambda^2_{\text{cutoff}}} + \cdots$$

$$= \frac{g^2(q^2)}{4\pi} = \frac{1}{1 + \beta_0 \ln q^2/\Lambda^2_{QCD}}$$




Asymptotic freedom means the short-distance (or high energy) domain of QCD is controllable by perturbation theory.

The flip side is that at long distances, $g^2$ becomes large — perturbation theory predicts its own demise.

Desirable — offers the chance that non-perturbative QCD confines

⋮

Only in exceptional circumstances can one predict strong interactions without non-perturbative methods.

(The great success of perturbative QCD — jet cross sections that agree over many orders of magnitude — use the operator product expansion & the renormalization group to get proton structure functions from DIS. Then used in $\frac{d\sigma}{dp_T}$ ts.)

The functional integral

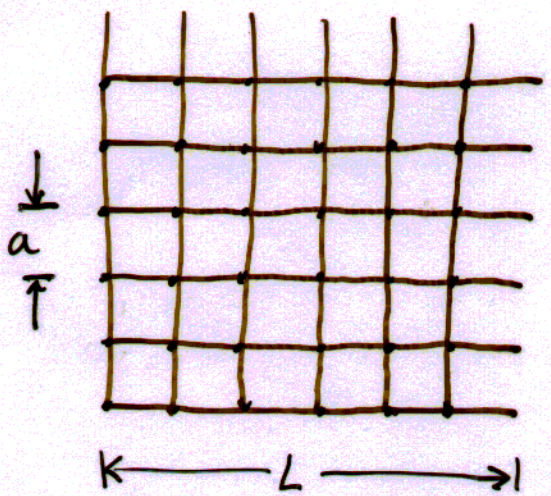
$$\langle \bullet \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q} \;\bullet\; e^{iS}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\; e^{iS}}$$

needs a non-perturbative meaning.

Noting the formal similarity to stat. mech

$$\langle \cdot \rangle = \frac{\sum_{\text{configs}} \cdot \; e^{-\beta H}}{\sum_{\text{configs}} e^{-\beta H}}$$

Ken Wilson (1974) realized that you could raid the toolbox of stat. mech. by putting the quarks and gluons on a lattice.




$L = N_s a$

For non-gauge quantum field theories, this is completely straightforward — derivatives are replaced with difference operators

To obtain a gauge invariant discretization requires a little math.

$$q(x) \longmapsto g(x)q(x) \qquad \bar{q}(x) \mapsto \bar{q}(x)\, g^{-1}(x)$$

$$g(x) \in SU(N) \qquad g = e^{i\lambda^a \omega^a/2}$$

"quarks transform covariantly"

$$A_\mu(x) \mapsto g(x)\, A_\mu\, g^{-1}(x) + g(x)\, \partial_\mu g^{-1}(x)$$

$$A_\mu = \tfrac{i}{2}\lambda^a A^a_\mu$$

"gluons transform like a connection"

$$D_\mu = \partial_\mu + A_\mu$$

$$D_\mu \longmapsto g(x)\, D_\mu\, g(x)^{-1}$$

What is the discrete version of $D_\mu$ ?

$$\frac{1}{a}\left[U_\mu(x)\,\psi(x+a\hat{e}_\mu) - \psi(x)\right]$$

$$U_\mu(x) = P \exp\int_0^a ds\, A_\mu(x+s\hat{e}_\mu)$$

$$U_\mu(x) \mapsto g(x)\, U_\mu(x)\, g^{-1}(x+a\hat{e}_\mu)$$

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$A_\mu = \frac{i}{2}\lambda^a A^a_\mu$ matter fields (quarks, Higgs, ...)

"gluons transform like ~~fire~~connections"

$D_\mu = \partial_\mu + A_\mu$ gluons live on links as $U_\mu(x)$

$$D_\mu \mapsto g(x)\, D_\mu\, g(x)^{-1}$$

What is the discrete version of $D_\mu$ ?

$$\frac{1}{a}\left[U_\mu(x)\,\psi(x+a\hat{e}_\mu) - \psi(x)\right]$$

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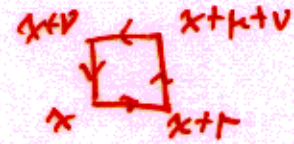
$$U_\mu(x) \mapsto g(x)\, U_\mu(x)\, g^{-1}(x+a\hat{e}_\mu)$$

So a simple Lagrangian is

$$\mathcal{L}_W = \mathcal{L}_{Wg} + \mathcal{L}_{Wq}$$




$$\mathcal{L}_{Wg} = \frac{1}{2g_0^2} \mathrm{Re}\,\mathrm{tr}\left[ U_\mu(x) U_\nu(x+a\hat{e}_\mu) U_\mu^\dagger(x+a\hat{e}_\nu) U_\nu^\dagger(x) - 1 \right]$$

$$\mathcal{L}_{Wq} = -m_0 \bar{\psi}\psi + \sum_\mu \bar{\psi} \frac{1-\gamma_\mu}{2} D_\mu^+ \psi \qquad \text{quarks fwd}$$

$$\sum_\mu \bar{\psi} \frac{1+\gamma_\mu}{2} D_\mu^- \psi \qquad \text{antiquarks bkw}$$

$$D_\mu^+ \psi(x) = \frac{1}{a}\left[ U_\mu(x)\psi(x+a\hat{e}_\mu) - \psi(x) \right]$$

$$D_\mu^- \psi(x) = \frac{1}{a}\left[ \psi(x) - U_\mu^\dagger(x-a\hat{e}_\mu)\psi(x-a\hat{e}_\mu) \right]$$

It is a relatively easy exercise to show that $\mathcal{L}_W \to \mathcal{L}_{QCD}$ if $a \to 0$ with bare couplings $g_0^2$, $m_0$ fixed.

Actually want $a \to 0$ with renormalized couplings fixed, for a quantum field theory.

Mathematical physicists believe that lattice gauge theory is a rigorous definition of QCD.

Trace out sequence of lattices with $a \to 0$ and mass ratios held fixed.

"Believe" means a proof is lacking. For QCD it is very plausible, owing to asymptotic freedom.

Lattice gauge theory can be attacked with many tools

weak coupling expansions

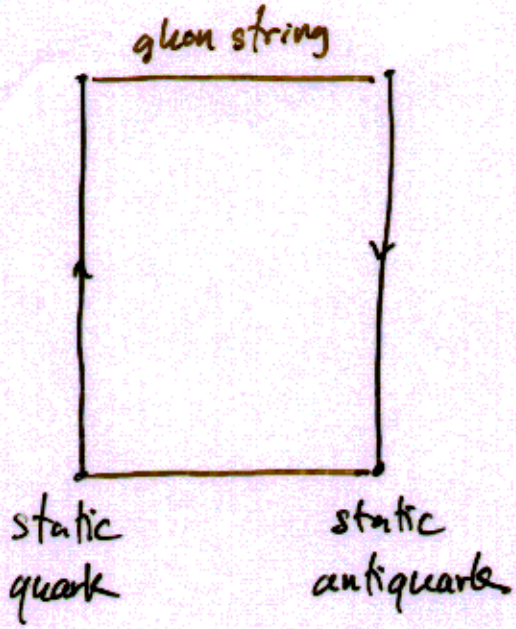
strong coupling expansions

Monte Carlo integration

The last two are non-perturbative.

There is one strong coupling calculation that everyone should know about

$R \times T$

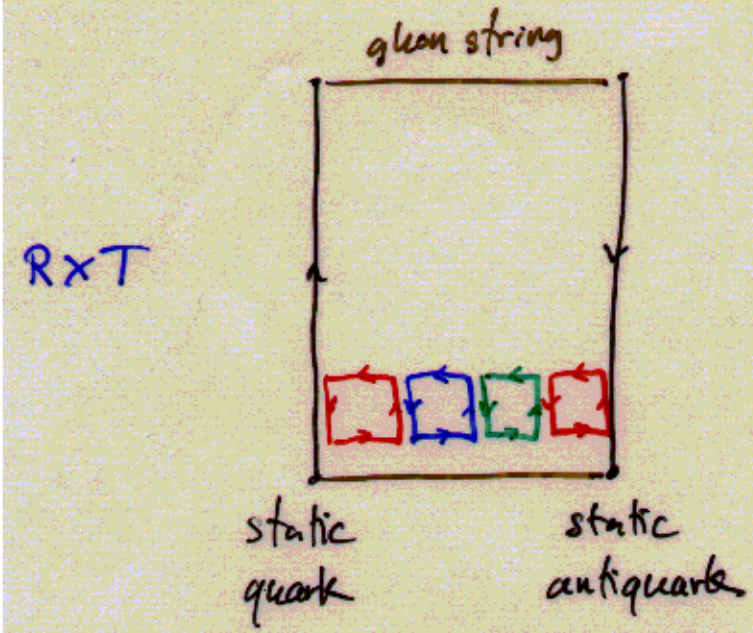



$$= P \exp \int_{\square} dz \cdot A(z)$$

$$= e^{-TV(R,T)}$$

Strong coupling treats $\frac{1}{g^2} F_{\mu\nu}^2$ as small

There is one strong coupling calculation that everyone should know about




$$= P \exp \int_{\square} dz \cdot A(z)$$

$$= e^{-TV(R,T)}$$

Strong coupling treats $\frac{1}{g^2}$ as small

Gauge invariance $\int dU = 1$

$$\int dU \, U = 0$$

so need $U$ and $U^+$, else get $0$

$$e^{-TV(R,T)} \sim \left(\frac{1}{g_0^2}\right)^{RT}$$ area law

$$V = R \ln g_0^2$$ confining potential

The problem with strong coupling is that it is the opposite from the continuum limit

$$a \to 0 \quad \Rightarrow \quad \frac{g_0^2}{4\pi} = \frac{1}{\beta_0 \ln a^2 \Lambda_{QCD}^2} \to 0$$

so it is not quantitative.

Nevertheless, it is amazing that confinement emerges so easily.

By the way, numerical calculations of large Wilson loops show that confining potential survives as $g_0^2$ is made smaller and smaller.

For quantitative calculations of hadronic matrix elements, strong coupling is not good enough. The tool of choice has become numerical integration of the functional integral, by a Monte Carlo method (MDP on Weds).

This method has taken a long time to mature, because — like any numerical technique — there are uncertainties associated with approximations, choices, etc.

Theorists were not raised on error bars — in addition to solving theoretical + computational problems, we have had to deal with this.

WHERE do the errors come from?

QCD is not a one scale problem, at least when you are trying to solve real problems

$$m_q \ll \Lambda_{QCD} \ll m_Q$$

The QCD scale $\Lambda_{QCD}$ can be anything from $\Lambda_{\overline{MS}} \approx 250$ MeV, $m_\rho = 770$ MeV upto

$$\frac{m_K^2}{m_s} = 2.5 \text{ GeV}$$

Light quarks have much smaller masses

$$m_s \approx 100 \text{ MeV} \qquad \hat{m} = \tfrac{1}{2}(m_u + m_d) = \frac{m_s}{26}$$

Heavy quarks have large masses

$$m_c \approx 1.3 \text{ GeV} \qquad m_b = 4 \text{ GeV}$$

$$m_t = 175 \text{ GeV}$$

Numerical calculations introduce two more scales, an ultraviolet cutoff $\frac{\pi}{a}$ and an infrared cutoff $L^{-1}$ $(L = N_s a)$.

Ideally

$$L^{-1} \ll m_q \ll \Lambda \ll m_Q \ll a^{-1}$$

but computer resources limit $N_s = L/a$ to $\lesssim 32$.

memory $\propto N_s^4 = (L/a)^4$

gauge CPU $\propto a^{-(4+z)}$

$z = 1$ or $2$

quark CPU $\propto (m_q a)^{-p}$

$p = 1-3$

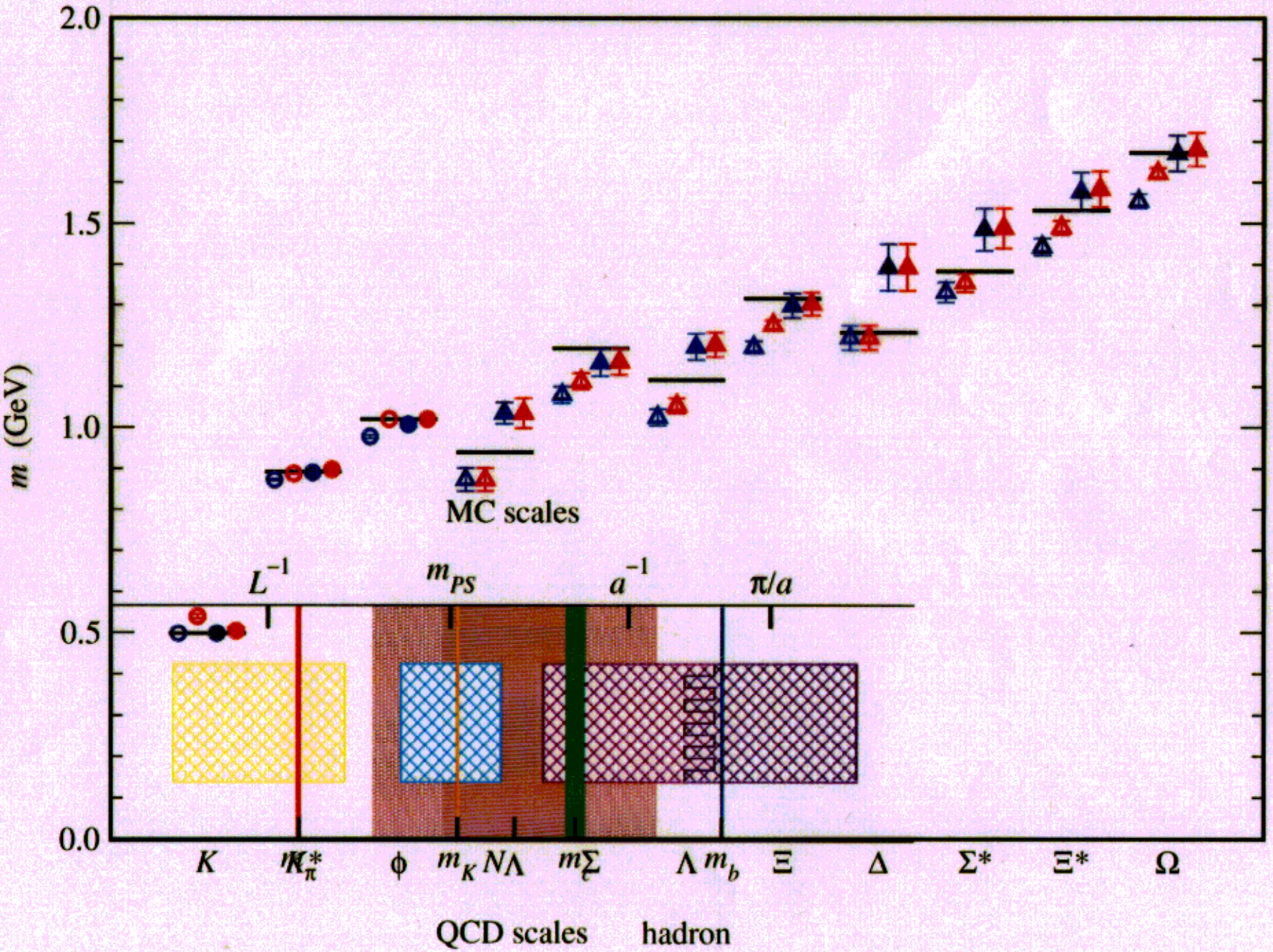
Part of the problem lies in the algorithms, but most of it lies in the fact that we live in 4-d spacetime

Consequently, the computer has the hierarchy

$$L^{-1} \lesssim m_q \lesssim \Lambda \ll m_Q \sim a^{-1}$$

The task is to get from numerical data computed here to the real world.

Fortunately, in each case we have an effective field theory which controls the extrapolation.


2.0
1.5
1.0
0.5
0.0
m (GeV)
MC scales
$L^{-1}$
$m_{PS}$
$a^{-1}$
$\pi/a$
K
$m_\pi$
$K^*$
$\phi$
$m_K$
N
$\Lambda$
$m_\Sigma$
$\Lambda$
$m_b$
$\Xi$
$\Delta$
$\Sigma^*$
$\Xi^*$
$\Omega$
QCD scales
hadron